6.003 Homework #1 Solutions

Problems

1. Solving differential equations

Solve the following differential equation

$$y(t) + 3\frac{dy(t)}{dt} + 2\frac{d^2y(t)}{dt^2} = 1$$

for $t \geq 0$ assuming the initial conditions $y(0) = 1$ and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 2$. Express the solution in closed form. Enter your closed form expression the the box below.

[Hint: assume the homogeneous solution has the form $Ae^{s_1 t} + Be^{s_2 t}$.]

$y(t) = \boxed{-4e^{-t} + 4e^{-t/2} + 1}$

First solve the homogeneous equation: $y_h(t) + 3\dot{y}_h(t) + 2\ddot{y}_h(t) = 0$. Assume $y_h(t) = Ae^{st}$. Then $\dot{y}_h(t) = sAe^{st}$ and $\ddot{y}_h(t) = s^2Ae^{st}$. Substitute into the homogeneous differential equation to obtain $(1 + 3s + 2s^2)Ae^{st} = 0$. Since e^{st} is never equal to zero, either A must be 0 or $1 + 3s + 2s^2$ must be zero. If A were zero, then the solution would be trivial (i.e., $y_h(t) = 0$), so the latter must be true to get a non-zero solution. From the factored form $(1 + s)(1 + 2s) = 0$, it is clear that s could be -1 or -0.5. Therefore the complete homogeneous solution could be written as

$$y_h(t) = Ae^{-t} + Be^{-t/2}$$

as in the hint. The particular solution has the same form as the inhomogenous part, so that $y_p(t) = 1$. To satisfy the initial conditions, we require that $y(t)$ (the sum of the homogeneous and particular parts) satisfies $y(0) = A+B+1 = 1$ and $\dot{y}(0) = -A-B/2 = 2$ so that $A = -4$ and $B = 4$. The final solution is

$$y(t) = -4e^{-t} + 4e^{-t/2} + 1\,.$$

2. Solving difference equations

Solve the following difference equation

$$8y[n] - 6y[n-1] + y[n-2] = 1$$

for $n \geq 0$ assuming the initial conditions $y[0] = 1$ and $y[-1] = 2$. Express the solution in closed form. Enter your closed form expression the the box below.

[Hint: assume the homogeneous solution has the form $Az_1^n + Bz_2^n$.]

$$y[n] = \boxed{\frac{1}{6}\left(\frac{1}{4}\right)^n + \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{3}}$$

First solve the homogeneous system: $8y_h[n] - 6y_h[n-1] + y_h[n-2] = 0$. Assume $y_h[n] = Az^n$. Then $y_h[n-1] = Az^{n-1} = z^{-1}Az^n$ and $y_h[n-2] = Az^{n-2} = z^{-2}Az^n$. Substitute into the original difference equation to obtain $(8 - 6z^{-1} + z^{-2})Az^n = 0$. Since z^n is never equal to zero, either A must be 0 or $(8 - 6z^{-1} + z^{-2})$ must be zero. If A were zero, then the solution would be trivial (i.e., $y_h[n] = 0$), so the latter must be true to get a non-zero solution. From the factored form $(4 - z^{-1})(2 - z^{-1}) = 0$, it is clear that z^{-1} could be 4 or 2. Therefore the complete homogeneous solution could be written as

$$y_h[n] = A\left(\frac{1}{4}\right)^n + B\left(\frac{1}{2}\right)^n$$

as in the hint. The particular solution has the same form as the non-homogeneous part, so that $y_p[n] = \frac{1}{3}$. To satisfy the initial conditions, we require $y[n]$ (which is the sum of the homogeneous and particular parts) satisfies $y[0] = A + B + 1 = \frac{1}{3}$ and $y[-1] = A/4 + B/2 + \frac{1}{3} = 2$ so that $A = \frac{1}{6}$ and $B = \frac{1}{2}$. The final solution is

$$y[n] = \frac{1}{6}\left(\frac{1}{4}\right)^n + \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{3}$$

3. Geometric sums

a. Expand $\dfrac{1}{1-a}$ in a power series.

power series: $1 + a + a^2 + a^3 + \cdots$

For what range of a does your answer converge?

range: $|a| < 1$

One can expand $\frac{1}{1-a}$ using synthetic division as follows:

$$\begin{array}{r|l}
 & 1 \;\; +a \;\; +a^2 \;\; +a^3 \;\; +\cdots \\ \hline
1-a & 1 \\
 & \underline{1 \;\; -a} \\
 & \quad a \\
 & \quad \underline{a \;\; -a^2} \\
 & \qquad a^2 \\
 & \qquad \underline{a^2 \;\; -a^3} \\
 & \qquad\quad a^3 \\
 & \qquad\quad \underline{a^3 \;\; -a^4} \\
 & \qquad\qquad \cdots
\end{array}$$

Alternatively, one could use a Taylor series:

$$\frac{d}{da}(1-a)^{-1} = (1-a)^{-2}$$
$$\frac{d^2}{da^2}(1-a)^{-1} = 2(1-a)^{-3}$$
$$\frac{d^3}{da^3}(1-a)^{-1} = 6(1-a)^{-4}$$
$$\frac{d^4}{da^4}(1-a)^{-1} = 24(1-a)^{-5}$$
$$\cdots$$

$$\begin{aligned}
\frac{1}{1-a} &= (1-a)^{-1}\Big|_{a=0} + \frac{d}{da}(1-a)^{-1}\bigg|_{a=0} a + \frac{1}{2}\frac{d^2}{da^2}(1-a)^{-1}\bigg|_{a=0} a^2 \\
&\quad + \frac{1}{3!}\frac{d^3}{da^3}(1-a)^{-1}\bigg|_{a=0} a^3 + \frac{1}{4!}\frac{d^4}{da^4}(1-a)^{-1}\bigg|_{a=0} a^4 + \cdots \\
&= 1 + a + a^2 + a^3 + a^4 + \cdots
\end{aligned}$$

These expressions converge iff $|a^n|$ tends toward zero, i.e., $|a| < 1$.

b. Express $\sum_{n=0}^{N-1} a^n$ in closed form.

closed form: $\boxed{\dfrac{1-a^N}{1-a}}$

For what range of a does your answer converge?

range: $\boxed{|a| < \infty}$

Let

$$y = \sum_{n=0}^{N-1} a^n.$$

Then

$$ay = a\sum_{n=0}^{N-1} a^n = \sum_{n=1}^{N} a^n.$$

If $N \geq 2$,

$$y - ay = (1-a)\,y = \sum_{n=0}^{N-1} a^n - \sum_{n=1}^{N} a^n = \left(1 + \sum_{n=1}^{N-1} a^n\right) - \left(\sum_{n=1}^{N-1} a^n + a^N\right) = 1 - a^N.$$

If $N = 1$, $y = 1$, so $(1-a)\,y = 1 - a^1$. If $N = 0$, $y = 0$, so $(1-a)\,y = 1 - a^0$. Therefore

$$(1-a)\,y = 1 - a^N$$

for all $N \geq 0$. Now if $a \neq 1$ we can divide both sides of the previous equation by $1-a$ to obtain a closed form result:

$$y = \sum_{n=0}^{N-1} = \frac{1-a^N}{1-a}.$$

If $a = 1$, then the closed form is indeterminant, but the limit $a \to 1$ can still be calculated using l'Hopital's rule,

$$y = \lim_{y \to 1} \frac{1-a^N}{1-a} = \lim_{y \to 1} \frac{-Na^{N-1}}{-1} = N.$$

Thus, the closed form can be taken for all finite a.

c. Expand $\dfrac{1}{(1-a)^2}$ in a power series.

power series: $1 + 2a + 3a^2 + 4a^3 + \cdots$

For what range of a does your answer converge?

range: $|a| < 1$

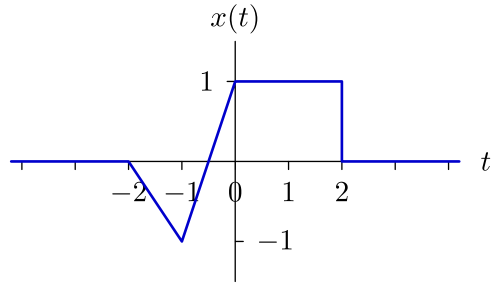
One can expand $\dfrac{1}{(1-a)^2}$ using synthetic division as follows:

$$
\begin{array}{r|llllll}
 & 1 & +2a & +3a^2 & +4a^3 & +\cdots \\ \cline{2-6}
1-2a+a^2 & 1 & & & & \\
 & 1 & -2a & +a^2 & & \\ \cline{2-4}
 & & +2a & -a^2 & & \\
 & & +2a & -4a^2 & +2a^3 & \\ \cline{3-5}
 & & & +3a^2 & -2a^3 & \\
 & & & +3a^2 & -6a^3 & +3a^4 \\ \cline{4-6}
 & & & & +4a^3 & -3a^4
\end{array}
$$

Alternatively, one could use a Taylor series.
The expansion converges as long as $\lim_{n\to\infty}(n+1)a^n = 0$, i.e., if $|a| < 1$.

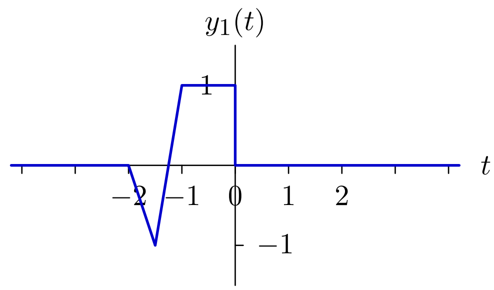
4. CT transformations

Let $x(t)$ represent the signal shown in the following plot.



The signal is zero outside the range $-2 < t < 2$.

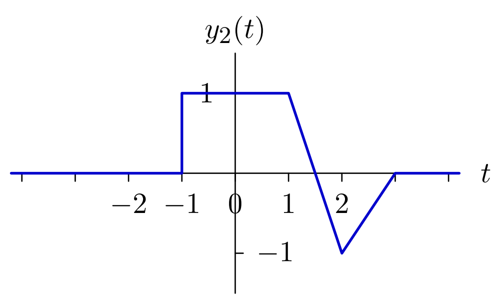
a. The following plot shows $y_1(t)$, which is a signal that is derived from $x(t)$.



Determine an expression for $y_1(t)$ in terms of $x(\cdot)$.

$y_1(t) =$ $x(2t+2)$

b. The following plot shows $y_2(t)$, which is a signal that is derived from $x(t)$.



Determine an expression for $y_2(t)$ in terms of $x(\cdot)$.

$y_2(t) =$ $x(1-t)$

c. Let $y_3(t) = x(2t+3)$. Determine all values of t for which $y_3(t) = 1$.

range of t : $-\frac{3}{2} \leq t < -\frac{1}{2}$

$x(t) = 1$ for $0 \leq t < 2$. Therefore $y_3(t) = 1$ if $0 < 2t+3 < 2$, i.e.,

$$-\frac{3}{2} \leq t < -\frac{1}{2}\,.$$

d. Assume that $x(t)$ can be written as the sum of an even part

$$x_e(t) = x_e(-t)$$

and an odd part

$$x_o(t) = -x_o(-t)\,.$$

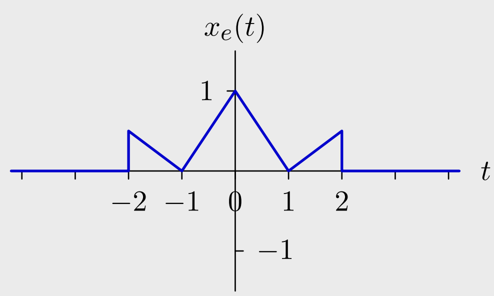
For what values of t is $x_e(t) = 0$?

values of t: $|t| \geq 2$ or $|t| = 1$

Let $x(t) = x_e(t) + x_o(t)$. Then $x(-t) = x_e(-t) + x_o(-t)$. By the definitions of even and odd, it follows that $x(-t) = x_e(t) - x_o(t)$. Add this to the first equation to get $x(t) + x(-t) = 2x_e(t)$. Thus

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

is uniquely determined by $x(t)$. The function $x_e(t)$ is plotted below.



From the plot, it is clear that $x_e(t) = 0$ if $|t| > 2$ or $|t| = 1$. Since $x(t)$ is defined to be zero at $t = \pm 2$, we should include those points as well, so $|t| \geq 2$ or $|t| = 1$.

Engineering Design Problems

5. Decomposing Signals

The even and odd parts of a signal $x[n]$ are defined by the following:

- $x_e[-n] = x_e[n]$ (i.e., x_e is an even function of n)
- $x_o[-n] = -x_o[n]$ (i.e., x_o is an odd function of n)
- $x[n] = x_e[n] + x_o[n]$

Let $x_r[n]$ represent the part of $x[n]$ that occurs for $n \geq 0$,

$$x_r[n] = \begin{cases} x[n] & n \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

Let $x_l[n]$ represent the part of $x[n]$ that occurs for $n < 0$),

$$x_l[n] = \begin{cases} x[n] & n < 0 \\ 0 & \text{otherwise} \end{cases} .$$

Notice that $x_r[0] = x[0]$ while $x_l[0] = 0$.

a. Is it possible to determine $x[n]$ (for all n) from $x_e[n]$ and $x_r[n]$?

Yes or **No**: Yes

If yes, explain a procedure for doing so. If no, explain why not.

We can find an expression for $x_o[n]$ in two parts. First, for $n \geq 0$, $x[n] = x_r[n]$. Since $x_o[n] = x[n] - x_e[n]$, it follows that $x_o[n] = x_r[n] - x_e[n]$ for $n \geq 0$. Second, for $n < 0$, we can use the fact that $x_o[n]$ is always $-x_o[-n]$ to find that $x_o[n] = -x_r[-n] + x_e[-n]$ for $n < 0$. Having constructed $x_o[n]$ for all n, it is easy to reconstruct $x[n]$ from the sum of $x_o[n]$ and $x_e[n]$ (which was given).

b. Is it possible to determine $x[n]$ (for all n) from $x_o[n]$ and $x_l[n]$?

Yes or **No**: No

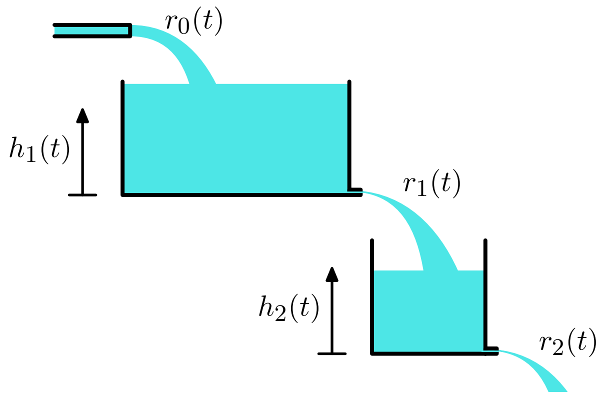
If yes, explain a procedure for doing so. If no, explain why not.

It is impossible to determine $x[0]$ from the information given, since $x_o[0] = 0$ and $x_l[0] = 0$. Therefore, it is impossible to reconstruct $x[n]$ from $x_o[n]$ and $x_l[n]$.

6. Leaky tanks

The following figure illustrates a cascaded system of two water tanks. Water flows

– into the first tank at a rate $r_0(t)$,
– out of the first tank and into the second at a rate $r_1(t)$, and
– out of the second tank at a rate $r_2(t)$.



The rate of flow out of each tank is proportional to the height of the water in that tank: $r_1(t) = k_1 h_1(t)$ and $r_2(t) = k_2 h_2(t)$, where k_1 and k_2 are each $0.2\,\mathrm{m}^2$/second. Both tanks have heights of $1\,\mathrm{m}$. The cross-sectional area of tank 1 is $A_1 = 4\,\mathrm{m}^2$ and that of the second tank is $A_2 = 2\,\mathrm{m}^2$. At time $t = 0$, both tanks are empty.

Part a. Let $x(t) = r_0(t)$ represent the input of the tank system and $y(t) = r_2(t)$ represent the output. Determine the relation between the input and the output. Express this relation as a differential equation of the form

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + a_2 \frac{d^2y(t)}{dt^2} + \cdots = x(t) + b_1 \frac{dx(t)}{dt} + b_2 \frac{d^2x(t)}{dt^2} + \cdots$$

where the coefficient of $x(t)$ is 1.

$a_0, a_1, a_2, \cdots$: $\boxed{1,\ 30,\ 200,\ 0,\ 0,\ \cdots}$

$b_1, b_2, b_3, \cdots$: $\boxed{0,\ 0,\ 0,\ \cdots}$

The volume of water in tank 1 is $A_1 h_1(t)$. Therefore, the time rate of change of water in tank 1 is $A_1 \frac{dh_1(t)}{dt}$. Since water is neither created nor destroyed, the time rate of change of water in tank 1 is equal to the difference between the rate of water that enters ($r_1(t)$) and exits ($r_2(t)$),

$$A_1 \frac{dh_1(t)}{dt} = r_0(t) - r_1(t)\,.$$

Substituting $r_1(t)/k_1$ for $h_1(t)$ and rearranging yields

$$\frac{A_1}{k_1} \frac{dr_1(t)}{dt} + r_1(t) = r_0(t)\,.$$

A similar relation holds for tank 2,

$$\frac{A_2}{k_2} \frac{dr_2(t)}{dt} + r_2(t) = r_1(t)\,.$$

Substitute $r_1(t)$ from the last equation into the prior equation to obtain a relation between $x(t) = r_0(t)$ and $y(t) = r_2(t)$,

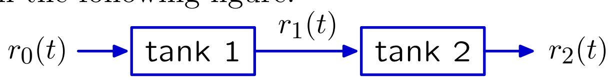
$$\frac{A_1 A_2}{k_1 k_2}\frac{d^2y(t)}{dt^2} + \quad \frac{A_1}{k_1} + \frac{A_2}{k_2} \quad \frac{dy(t)}{dt} + y(t) = 200\frac{d^2y(t)}{dt^2} + 30\frac{dy(t)}{dt} + y(t) = x(t)\,.$$

Part b. Assume that $r_0(t)$ is held constant at rate r_0. What is the maximum value of r_0 such that neither tank will ever overflow if both tanks start out empty.

r_0: $0.2\,\mathrm{m}^3/\mathrm{s}$

Initially, $h_1(t)$ is zero, so $r_1(t)$ is also zero. Thus, r_0 causes $h_1(t)$ to rise. As $h_1(t)$ rises, $r_1(t)$ increases until $r_1(t) = r_0$ or tank 1 overflows. Tank 1 will overflow when $r_1 = k_1 \times 1\,\mathrm{m}$. Therefore, to keep tank 1 from overflowing, we require that $r_0 = r_1 < k_1 \times 1\,\mathrm{m}$, i.e., $r_0 < 0.2\,\mathrm{m}^3/\mathrm{s}$. The same reasoning applies to tank 2. Therefore, the maximum value of r_0 such that neither tank overflows is $0.2\,\mathrm{m}^3/\mathrm{s}$.

Part c. Because $r_1(t)$ is both the output of the first tank and the input of the second tank, we can equivalently think of the two-tank system as a cascade of two one-tank systems, as shown in the following figure.



Determine a differential equation that relates $r_1(t)$ to $r_0(t)$. Determine the solution to this differential equation when $r_0(t)$ is held constant at $0.1\,\mathrm{m}^3/\mathrm{s}$. Assume that tank #1 is initially empty.

$r_1(t) =$ $0.1 - 0.1e^{-t/20}$

As in part a,

$$\frac{A_1}{k_1}\frac{dr_1(t)}{dt} + r_1(t) = r_0(t)\,.$$

Thus

$$20\frac{dr_1(t)}{dt} + r_1(t) = 0.1\,.$$

Assume a solution of the form

$$r_1(t) = Ae^{-t/\tau} + B$$

for $t > 0$. Substitute into the differential equation to obtain

$$-\frac{20A}{\tau} + A \quad e^{-t/\tau} + B = 0.1$$

Thus $B = 0.1$ and $\tau = 20$, so that

$$r_1(t) = Ae^{-t/20} + 0.1$$

for $t > 0$. But $r_1(0) = 0$. Therefore $A = -0.1$, and the final solution is

$$r_1(t) = 0.1 - 0.1e^{-t/20}\,.$$

The flow $r_1(t)$ starts at zero and exponentially approaches $0.1\,\mathrm{m}^3/\mathrm{s}$ with a time constant τ of 20 s.

Part d. We could similarly determine a differential equation that relates $r_2(t)$ to $r_1(t)$ and solve it for $r_2(t)$ given the solution for $r_1(t)$ given in Part c. As an alternative, we can use a numerical method.

Use the forward Euler approximation to generate a discrete approximation to the differential relation between $r_1(t)$ and $r_0(t)$, as follows. Let $r_0(t)$ and $r_1(t)$ be approximated by discrete sequences $r_0[n] = r_0(nT)$ and $r_1[n] = r_1(nT)$, where T represents the step size. Then approximate the continuous-time derivative at time nT by a first difference:

$$\left.\frac{dr_1(t)}{dt}\right|_{t=nT} \approx \frac{r_1[n+1] - r_1[n]}{T}.$$

Solve this difference equation for $r_1[n+1]$ in terms of values of $r_1[k]$ and $r_0[k]$ where $k < n+1$ and enter the result below.

$r_1[n+1] =$ $\boxed{r_1[n] + \frac{T}{20}\big(r_0[n] - r_1[n]\big)}$

$$20\frac{r_1[n+1] - r_1[n]}{T} + r_1[n] = r_0[n]$$

$$r_1[n+1] = r_1[n] + \frac{T}{20}\big(r_0[n] - r_1[n]\big)$$

Part e. Use your favorite computer language to solve this recursion for the special case when the input $r_0[n]$ is held constant at $0.1\,\mathrm{m}^3/\mathrm{s}$, tank #1 is initially empty, and $T =$ 1 second (see example code in box below). Make a plot of your solution for $0 < t < 60$. Also plot the analytic result from part c on the same axes. Determine the maximum difference between the analytic and numerical results.

maximum difference: $\boxed{< \ 0.00094}$

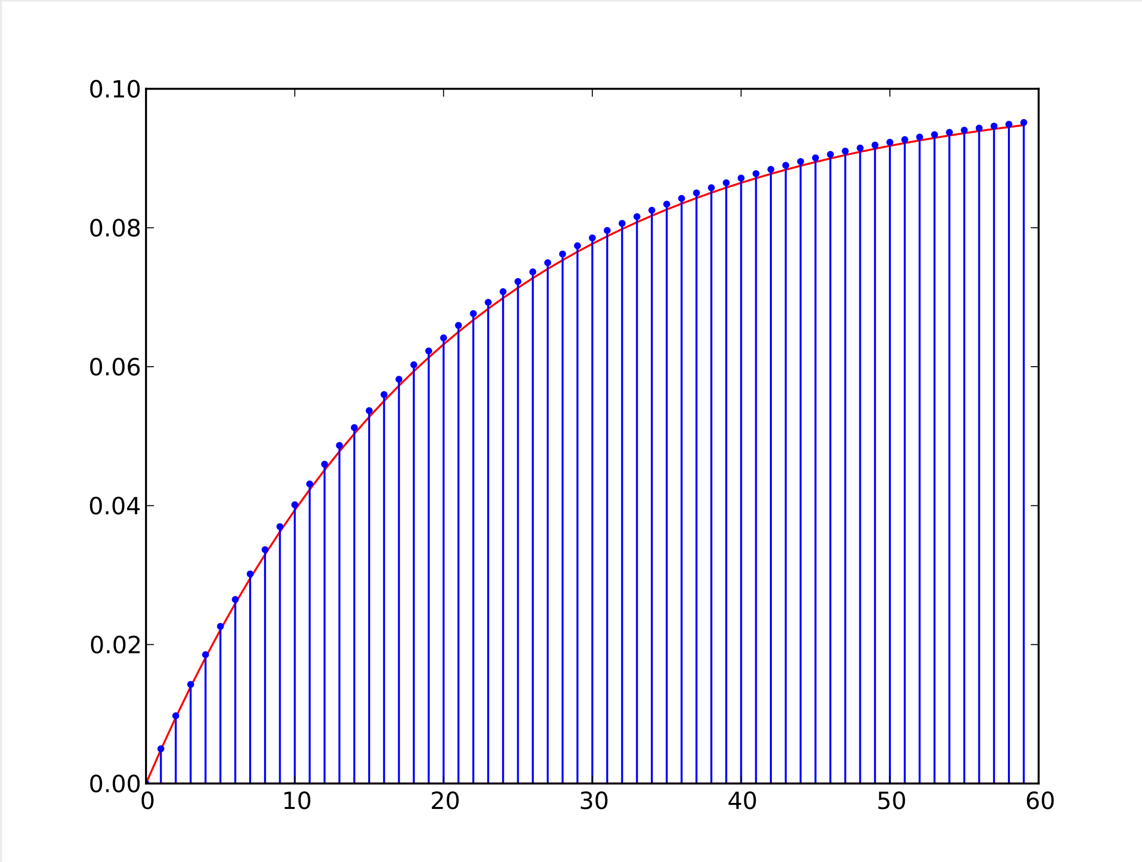
```
import math
from pylab import *

ya = []                        # analytic
for t in range(60):
    ya.append(0.1 - 0.1*math.e**(-t/20.))
print ya
plot(range(60),ya,'r-')

T = 1
yn = [0]                       # numerical
for i in range(1,60):
    yn.append(yn[i-1]+T/20.*(0.1-yn[i-1]))
print
print yn
stem(range(60),yn,'b-','b.','r-')

print
print max([yn[i]-ya[i] for i in range(60)])

show()
```



The biggest difference is less than 0.00094.

Part f. Modify your code to calculate numerical approximations to both $r_1(t)$ and $r_2(t)$. Plot results for both on the same axes. Explain similarities and differences of these two results for both small times and large times.

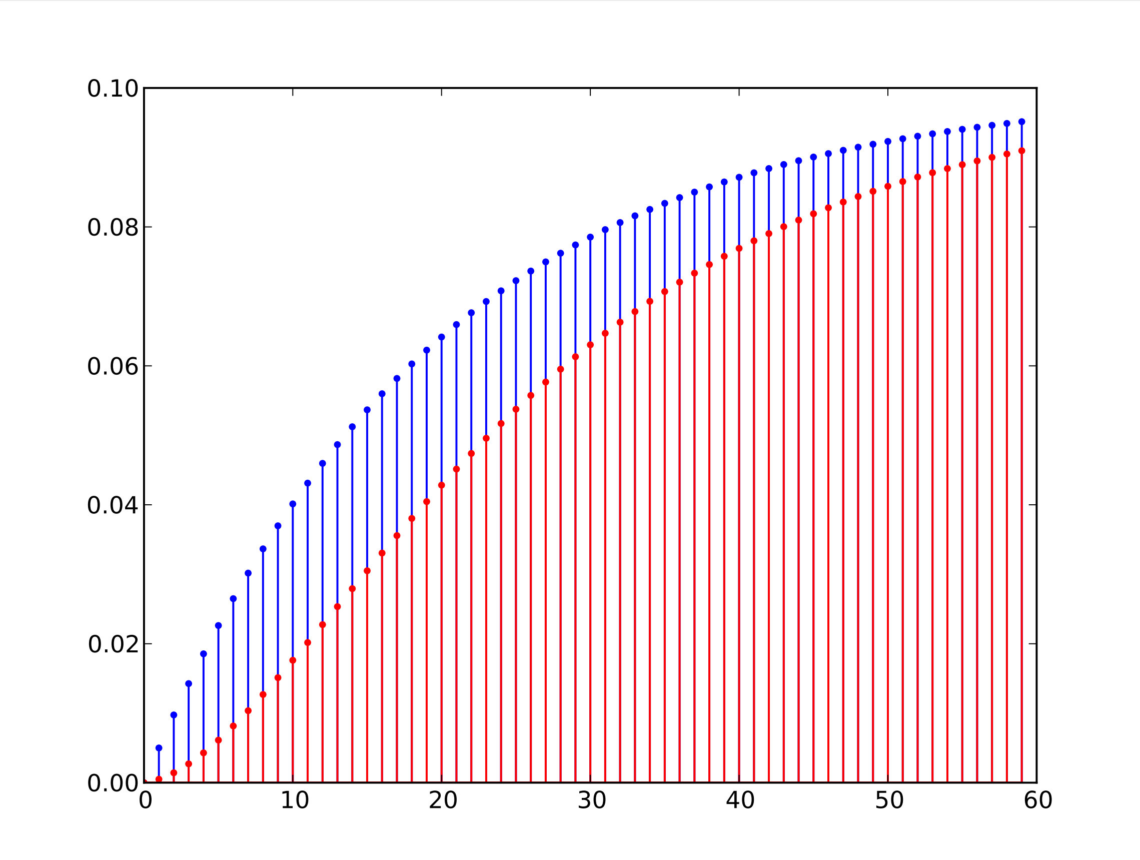
similarities: same initial value, same final value

differences: influx to tank 2 starts more slowly than influx to tank 1

```
import math
from pylab import *

T = 1
r1 = [0]                    # initial conditions
r2 = [0]
for i in range(1,60):
    r1.append(r1[i-1]+T/20.*(0.1-r1[i-1]))
    r2.append(r2[i-1]+T/10.*(r1[i]-r2[i-1]))
print
print r1
stem(range(60),r1,'b-','b.','r-')
stem(range(60),r2,'r-','r.','r-')
```

```
show()
```



Since the rate of influx to tank 2 starts more slowly than that to tank 1, the height in the second tank gets started more slowly than that in the first. Ultimately, the heights of water in both tanks approach the same value.

7. Drug dosing

When drugs are used to treat a medical condition, doctors often recommend starting with a higher dose on the first day than on subsequent days. In this problem, we consider a simple model to understand why. Assume that the human body is a tank of blood and that drugs instantly dissolve in the blood when ingested. Further assume that drug vanishes from the blood (either because it is broken down or because it is flushed by the kidneys) at a rate that is proportional to drug concentration.

Let $x[n]$ represent the amount of drug taken on day n, and let $y[n]$ represent the total amount of drug in the blood on day n, just after the dose $x[n]$ has dissolved in the blood, so that

$$y[n] = x[n] + \alpha y[n-1]\,.$$

a. Assume that no drug is in the blood before day 0, and that one unit of drug is taken each day, starting with day 0.

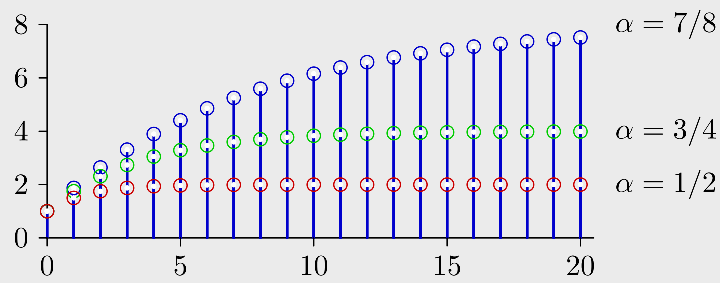
1. Determine an expression for the amount of drug in the blood immediately after the dose on day n has dissolved.

amount: $\boxed{\dfrac{1-\alpha^{n+1}}{1-\alpha}}$

Solve by iteration:

n	$y[n]$
0	1
1	$1+\alpha$
2	$1+\alpha+\alpha^2$
3	$1+\alpha+\alpha^2+\alpha^3$
$\cdots$	$\cdots$
n	$\dfrac{1-\alpha^{n+1}}{1-\alpha}$

2. Plot the amount of drug in the blood as a function of day number for $\alpha = \frac{1}{2}$, $\frac{3}{4}$, and $\frac{7}{8}$.



3. Determine an expression for the steady-state amount of drug in the blood, i.e., $\lim_{n\to\infty} y[n]$.

$\lim\limits_{n\to\infty} y[n]$: $\boxed{\dfrac{1}{1-\alpha}}$

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha}$$

$$\lim_{n\to\infty} y[n] = \frac{1}{1-\alpha}$$

b. In part a, the amount of drug in the blood ramps up over the first few days, before reaching a steady-state value. Suggest a different initial dose $x[0]$ that will result in a more constant amount of drug in the blood (with $x[n]$ remaining at 1 for all $n \geq 1$).

initial dose: $\boxed{\frac{1}{1-\alpha}}$

Consider what happens to the difference equation

$$y[n] = x[n] + \alpha y[n-1]$$

as $n \to \infty$. The value of y at time $n-1$, which is equal to the steady-state value, is transformed to y at time n, which is also equal to the steady-state value. It follows that if $x[0]$ were set equal to the steady-state amount of drug in the blood (i.e., $x[0] = \frac{1}{1-\alpha}$) then the system would behave as though it were in steady state from the outset. We can express this condition mathematically as follows.

$$y[0] = \frac{1}{1-\alpha}$$

$$y[1] = 1 + \alpha\frac{1}{1-\alpha} = \frac{1}{1-\alpha}$$

$$y[2] = 1 + \alpha\frac{1}{1-\alpha} = \frac{1}{1-\alpha}$$

$$\ldots$$

Thus $x[0]$ should be $\frac{1}{1-\alpha}$.